

Subleading Terms in the Collinear Limit of Yang-Mills Amplitudes

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Abstract

For two massless particles i and j , the collinear limit is a special kinematic configuration in which the particles propagate with parallel four-momentum vectors, with the total momentum P distributed as $p_i = xP$ and $p_j = (1-x)P$, so that $s_{ij} \equiv (p_i + p_j)^2 = P^2 = 0$. In Yang-Mills theory, if i and j are among N gauge bosons participating in a scattering process, it is well known that the partial amplitudes associated to the (single trace) group factors with adjacent i and j are singular in the collinear limit and factorize at the leading order into $(N-1)$ -particle amplitudes times the universal, x -dependent Altarelli-Parisi factors. We give a precise definition of the collinear limit and show that at the tree level, the subleading, non-singular terms are related to the amplitudes with a single graviton inserted instead of two collinear gauge bosons. To that end, we argue that in one-graviton Einstein-Yang-Mills amplitudes, the graviton with momentum P can be replaced by a pair of collinear gauge bosons carrying arbitrary momentum fractions xP and $(1-x)P$.

In high energy particle physics, collinear kinematics are very common. Viewed in the laboratory frame, all quarks and gluons (partons) propagating inside protons accelerated at the Large Hadron Collider (LHC) move in the beam direction, with a very little of transverse momentum. Since the early days of Quantum Chromodynamics (QCD), such collinear parton configurations have been in the focus of perturbative computations. In the so-called leading logarithmic approximation, the violation of Bjorken scaling [1] in proton structure functions can be understood as an effect of $1 \rightarrow 2$ parton decays which are necessarily collinear. They are described by Altarelli-Parisi probabilities [2] and involve the running gauge coupling constant that brings the fundamental QCD mass scale. When protons collide at high energies, many quarks and gluons are often produced in a single two-parton collision. Multi-parton amplitudes favor collinear final state configurations due to the singular behaviour that will be discussed below. At the LHC, such partons fragment into hadronic jets.

In general, for two massless particles i and j , the collinear limit is defined as a special kinematic configuration in which the particles propagate with parallel four-momentum vectors, with the total momentum P distributed as $p_i = xP$ and $p_j = (1-x)P$, so that $s_{ij} \equiv (p_i + p_j)^2 = P^2 = 0$. In QCD, as in any Yang-Mills theory, if i and j are among N gluons participating in a scattering process, it is well known that the partial amplitudes [3] associated to the (single trace) color factors with adjacent i and j are singular in the collinear limit and factorize at the leading order into $(N-1)$ -gluon amplitudes times the universal, x -dependent Altarelli-Parisi factors (three-gluon MHV amplitudes). The singularity is a simple pole at $s_{ij} = 0$ due to an intermediate gluon propagating on zero mass shell. In this paper, we go beyond the leading pole approximation and discuss non-factorizable, finite contributions¹.

In order to give a precise definition of the leading and subleading parts, we need to specify how the collinear limit is reached from a generic kinematic configuration. Let us specify to generic light-like momenta $p_i = p_{N-1}$, $p_j = p_N$ and introduce two light-like vectors P and r such that the momentum spinors decompose as

$$\begin{aligned} \lambda_{N-1} &= \lambda_P \cos \theta - \epsilon \lambda_r \sin \theta , & \tilde{\lambda}_{N-1} &= \tilde{\lambda}_P \cos \theta - \epsilon \tilde{\lambda}_r \sin \theta , \\ \lambda_N &= \lambda_P \sin \theta + \epsilon \lambda_r \cos \theta , & \tilde{\lambda}_N &= \tilde{\lambda}_P \sin \theta + \epsilon \tilde{\lambda}_r \cos \theta , \end{aligned} \tag{1}$$

hence

$$\begin{aligned} p_{N-1} &= \mathbf{c}^2 P - \epsilon \mathbf{sc}(\lambda_P \tilde{\lambda}_r + \lambda_r \tilde{\lambda}_P) + \epsilon^2 \mathbf{s}^2 r , \\ p_N &= \mathbf{s}^2 P + \epsilon \mathbf{sc}(\lambda_P \tilde{\lambda}_r + \lambda_r \tilde{\lambda}_P) + \epsilon^2 \mathbf{c}^2 r , \end{aligned} \tag{2}$$

¹ The “leading logarithms” come from integrating such poles. In the language of perturbative QCD, the subleading terms discussed here belong to so-called “higher twist” contributions.

where

$$\mathbf{c} \equiv \cos \theta = \sqrt{x} \ , \quad \mathbf{s} \equiv \sin \theta = \sqrt{1-x} \ . \quad (3)$$

We also have

$$\langle N-1 \ N \rangle = \epsilon \langle Pr \rangle \ , \quad [N-1 \ N] = \epsilon [Pr] \ . \quad (4)$$

The total momentum is:

$$p_{N-1} + p_N = P + \epsilon^2 r \ , \quad (p_{N-1} + p_N)^2 \equiv s_{N-1,N} = 2Pr \epsilon^2 \ . \quad (5)$$

The collinear configuration will be reached in the $\epsilon \rightarrow 0$ limit and the tree amplitudes discussed below will be expanded in powers of ϵ . Partial amplitudes with adjacent $N-1$ and N contain single (factorization) poles. Thus their leading terms are of order $\mathcal{O}(\epsilon^{-1})$ and the subleading ones are of order $\mathcal{O}(\epsilon^0)$. Collinear expansions of partial amplitudes with non-adjacent $N-1$ and N start at the subleading $\mathcal{O}(\epsilon^0)$ order.

The leading collinear behaviour of amplitudes with adjacent $N-1, N$ is well known [3] and depends on respective helicities. For identical helicities,

$$\begin{aligned} A(1, \dots, N-1^+, N^+) &= \frac{1}{\langle N-1 \ N \rangle_{\mathbf{sc}}} A(1, \dots, P^+) + \epsilon^0 \text{Sub}^{++} + \dots \ , \\ A(1, \dots, N-1^-, N^-) &= \frac{1}{[N-1 \ N]_{\mathbf{sc}}} A(1, \dots, P^-) + \epsilon^0 \text{Sub}^{--} + \dots \ , \end{aligned} \quad (6)$$

where we used superscripts to denote helicity states. Here, Sub denote subleading contributions which are the focus of this work. The remaining terms vanish in the $\epsilon \rightarrow 0$ limit. For opposite helicities:

$$\begin{aligned} A(1, \dots, N-1^+, N^-) &= \frac{\mathbf{s}^3}{\langle N-1 \ N \rangle_{\mathbf{c}}} A(1, \dots, P^-) + \frac{\mathbf{c}^3}{[N-1 \ N]_{\mathbf{s}}} A(1, \dots, P^+) \\ &\quad + \epsilon^0 \text{Sub}^{+-} + \dots \ . \end{aligned} \quad (7)$$

The starting point for our discussion of subleading terms is the recent observation [4] that the tree-level Einstein-Yang-Mills amplitudes describing decays of a single graviton or a dilaton into a number of gauge bosons, can be written as linear combinations of pure gauge amplitudes in which the graviton (or dilaton) is replaced by a pair of gauge bosons. Their ± 1 helicities add up to ± 2 for the graviton or to 0 for the dilaton. Each of them carries exactly one half of the original graviton or dilaton momentum, which is a special case of a collinear configuration with $\mathbf{s} = \mathbf{c} = \sqrt{1/2}$. From now on we will focus on graviton amplitudes. The crucial point is that the relations derived in [4] can be extended

to *arbitrary* collinear configurations, in the following way

$$\begin{aligned}
A_{\text{EYM}}(1, 2, \dots, N-2; P^{\pm 2}) = & \quad (8) \\
= \frac{\kappa \mathbf{s}^2}{g^2} \left\{ \sum_{l=2}^{\lceil \frac{N}{2} \rceil - 1} \sum_{i=2}^l \left(\sum_{j=i}^l s_{j, N-1} \right) A_{\text{YM}}(1, \dots, i-1, N^{\pm}, i, \dots, l, N-1^{\pm}, l+1, \dots, N-2) \right. \\
& \left. + \sum_{l=\lceil \frac{N}{2} \rceil}^{N-3} \sum_{i=l+1}^{N-2} \left(\sum_{j=l+1}^i s_{j, N-1} \right) A_{\text{YM}}(1, \dots, l, N-1^{\pm}, l+1, \dots, i, N^{\pm}, i+1, \dots, N-2) \right\},
\end{aligned}$$

where κ and g are the gravitational and gauge coupling constants, respectively². On the left hand side, we have a mixed gauge-gravitational amplitude involving a single graviton of momentum P , helicity ± 2 as indicated by the superscript, and $N-2$ gluons. This amplitude is associated to a single trace color factor with the respective gluon ordering. On the right hand side, we have a linear combination of pure gauge, partial amplitudes weighted by the kinematic invariants $s_{j, N-1} = 2p_j p_{N-1}$. Here, the graviton is replaced by two gluons in the collinear configuration:

$$p_{N-1} = \mathbf{c}^2 P = xP, \quad p_N = \mathbf{s}^2 P = (1-x)P, \quad (9)$$

i.e. the leading $\mathcal{O}(\epsilon^0)$ order of Eqs. (1) and (2). Note that on the right hand side, $N-1$ and N are never adjacent, therefore the Einstein-Yang-Mills amplitude emerges from the collinear limit of Yang-Mills amplitudes at the subleading $\mathcal{O}(\epsilon^0)$ order. In order to further discuss Eq. (8), it is useful to write it explicitly for $N = 5, 6, 7$:

$$A(1, 2, 3; P^{\pm 2}) = \frac{\kappa \mathbf{s}^2}{g^2} s_{24} A(1, 5^{\pm}, 2, 4^{\pm}, 3), \quad (10)$$

$$A(1, 2, 3, 4; P^{\pm 2}) = \frac{\kappa \mathbf{s}^2}{g^2} \left\{ s_{25} A(1, 6^{\pm}, 2, 5^{\pm}, 3, 4) + s_{45} A(1, 2, 3, 5^{\pm}, 4, 6^{\pm}) \right\}, \quad (11)$$

$$\begin{aligned}
A(1, 2, 3, 4, 5; P^{\pm 2}) = & \frac{\kappa \mathbf{s}^2}{g^2} \left\{ s_{26} A(1, 7^{\pm}, 2, 6^{\pm}, 3, 4, 5) + s_{36} A(1, 2, 7^{\pm}, 3, 6^{\pm}, 4, 5) \right. \\
& \left. + (s_{36} + s_{26}) A(1, 7^{\pm}, 2, 3, 6^{\pm}, 4, 5) + s_{56} A(1, 2, 3, 4, 6^{\pm}, 5, 7^{\pm}) \right\}. \quad (12)
\end{aligned}$$

The fact that the relations written in Ref. [4] can be extended from $\mathbf{s} = \mathbf{c} = \sqrt{1/2}$, *i.e.* from $x = 1/2$, to an arbitrary collinear configuration by inserting a simple factor of $\mathbf{s}^2 = 1 - x$ is highly non-trivial. It is easiest to check for the helicity configurations described by MHV amplitudes [5] on the r.h.s. of Eq. (8), *i.e.* when the collinear pair is

² $\lceil \frac{N}{2} \rceil$ is the smallest integer greater than or equal to $\frac{N}{2}$. Since the graviton is identified by its momentum P , we can skip in the following the EYM and YM labelings of the amplitudes.

among $N - 2$ gluons with identical helicities and there are only two gluons with opposite helicities³. Then the r.h.s. of Eq. (8) is a homogenous function of spinor (and momentum) variables and it is easy to see that, in this case, arbitrary value of x can be reached from $x = 1/2$ by a simple rescaling of the amplitudes, with the net effect of an overall $1 - x$ factor. For other helicity configurations, the amplitudes are not homogenous in the momenta of collinear gluons. Already at the NMHV level, individual amplitudes contain poles in three-gluon channels $[ijN-1]$ and $[ijN]$ (with $i, j \neq N-1, N$), characterized by the kinematic invariants

$$\begin{aligned} t_{ijN-1} &\equiv (p_i + p_j + p_{N-1})^2 = \mathbf{c}^2 t_{ijP} + \mathbf{s}^2 s_{ij} + \dots \\ t_{ijN} &\equiv (p_i + p_j + p_N)^2 = \mathbf{s}^2 t_{ijP} + \mathbf{c}^2 s_{ij} + \dots \end{aligned} \quad (13)$$

Such poles must cancel on the r.h.s. of Eq. (8) for the EYM amplitude to be free of unphysical singularities. In the appendix, we show that it is indeed the case for $N = 6$, and obtain an explicit expression for $A(1^+, 2^+, 3^-, 4^+; P^{-2})$ in agreement with Eq. (11). For $N = 7$, a similar check is still possible but it involves very tedious computations. Starting from $N = 8$, NNMHV amplitudes can appear on the r.h.s. of Eq. (8), therefore a complete proof would have to rely on more general representation of tree amplitudes or on recursion relations. Actually, the most straightforward way is to consider these amplitudes as a zero-slope limit of superstring disk amplitudes involving open and closed strings. Then, Eqs. (8) can be proven for arbitrary helicity configurations [6]. In this work however, we focus on field-theoretical amplitudes.

At the tree level, there are $(N - 3)!$ independent N -gluon amplitudes [7]. For a given N , we can express the Yang-Mills amplitudes appearing in Eq. (8) in terms of the basis $A(1, \sigma(2, 3, \dots, N-2), N-1, N)$, where σ denotes the set of $(N - 3)!$ permutations of $2, 3, \dots, N-2$. Let us start from $N = 5$, as in Eq. (10), where we use:

$$A(1, 5, 2, 4, 3) = \frac{s_{21}}{s_{25}} A(1, 2, 3, 4, 5) + \frac{s_{21} + s_{23}}{s_{25}} A(1, 3, 2, 4, 5). \quad (14)$$

As a result, we conclude that the following relation is valid up to the $\mathcal{O}(\epsilon^0)$ order:

$$s_{3P} A(1, 2, 3, 4^\pm, 5^\pm) - s_{2P} A(1, 3, 2, 4^\pm, 5^\pm) = \frac{g^2}{\kappa x} A(1, 2, 3; P^{\pm 2}). \quad (15)$$

By using Eq. (6) and BCJ relations for four-gluon amplitudes [7], it is easy to see that the leading collinear singularities $\mathcal{O}(\epsilon^{-1})$ drop out, therefore Eq. (15) connects the subleading

³ The other case, when the collinear pair carry helicities opposite to all other $N-2$ gluons, does not contribute because the corresponding amplitudes vanish in the collinear limit as ϵ^4 .

terms with the mixed gauge-gravitational amplitude. For $N = 5$, we obtain one relation between the subleading parts of two independent amplitudes. For $N = 6$, a similar equation reads:

$$\begin{aligned} s_{4P}A(1, 2, 3, 4, 5^\pm, 6^\pm) - s_{3P}[A(1, 2, 4, 3, 5^\pm, 6^\pm) + A(1, 4, 2, 3, 5^\pm, 6^\pm)] \\ + s_{2P}A(1, 4, 3, 2, 5^\pm, 6^\pm) = \frac{g^2}{\kappa x}A(1, 2, 3, 4; P^{\pm 2}) . \end{aligned} \quad (16)$$

In this case, however, we have two additional mixed amplitudes, say $A(1, 3, 2, 4; P^{\pm 2})$ and $A(1, 2, 4, 3; P^{\pm 2})$, that can be used in similar relations, obtained by interchanging $2 \leftrightarrow 3$ and $3 \leftrightarrow 4$, respectively. As a result, we obtain three relations for the subleading parts of six independent gauge amplitudes⁴. For $N = 7$,

$$\begin{aligned} s_{5P}A(1, 2, 3, 4, 5, 6^\pm, 7^\pm) \\ - s_{4P}[A(1, 2, 3, 5, 4, 6^\pm, 7^\pm) + A(1, 2, 5, 3, 4, 6^\pm, 7^\pm) + A(1, 5, 2, 3, 4, 6^\pm, 7^\pm)] \\ + s_{3P}[A(1, 5, 4, 2, 3, 6^\pm, 7^\pm) + A(1, 5, 2, 4, 3, 6^\pm, 7^\pm) + A(1, 2, 5, 4, 3, 6^\pm, 7^\pm)] \\ - s_{2P}A(1, 5, 4, 3, 2, 6^\pm, 7^\pm) = \frac{g^2}{\kappa x}A(1, 2, 3, 4, 5; P^{\pm 2}) . \end{aligned} \quad (17)$$

In this case, there are 24 independent Yang-Mills amplitudes with the subleading collinear behaviour constrained by twelve Einstein-Yang-Mills amplitudes. For arbitrary N a similar formula reads

$$\sum_{\rho \in \mathcal{P}_N} (-1)^{m_\rho} s_{\rho(N-2)P} A(1, \rho(2, \dots, N-2), N-1, N) = \frac{g^2}{\kappa x} A(1, \dots, N-2; P) , \quad (18)$$

where \mathcal{P}_N is a subset of permutations acting on $2, \dots, N-2$ and $m_\rho \in \{0, 1\}$ as specified in [6]. Now there are $(N-3)!/2$ independent constraints.

We see that the subleading collinear behaviour of pure gauge amplitudes is determined in part by the amplitudes with the graviton inserted instead of the collinear pair. Twice as many constraints are necessary, however, in order to fully determine the subleading terms for all amplitudes. In another physically interesting case of soft ($x \rightarrow 0$) divergences, the subleading behaviour has been recently discussed in Einstein's gravity [8] and in Yang-Mills theory [9]. We hope that similar considerations will allow complete determination of the subleading behaviour in the collinear case.

The fact that the graviton can be replaced by two gluons in arbitrary collinear configurations in the single-graviton amplitudes of Eq. (8) raises an interesting question whether

⁴ Three other mixed amplitudes are related by parity reflections, therefore they do not provide additional constraints.

pure Einstein, multi-graviton amplitudes share this property. The recent linearization [10] of KLT relations [11] suggests that this may be the case. It would be another indication for the existence of some underlying gauge structure in quantum gravity.

Appendix

We will show that Eq. (11) holds for $A(1^+, 2^+, 3^-, 4^+; P^{-2})$. To that end, we take the collinear limits, c.f. Eq. (1), of the six-gluon NMHV amplitudes written in Ref. [3]:

$$A(1^+, 6^-, 2^+, 5^-, 3^-, 4^+) = \frac{\langle 3P \rangle^4}{s_{1P}s_{2P}s_{3P}} \left\{ \frac{[14]^2[23]^2}{\mathbf{c}^2(\mathbf{s}^2 s_{23} + \mathbf{c}^2 s_{14})s_{14}} + \frac{[12]^2[34]^2}{\mathbf{s}^2(\mathbf{s}^2 s_{34} + \mathbf{c}^2 s_{12})s_{34}} + \frac{[12][23][34][41]}{\mathbf{c}^2 \mathbf{s}^2 s_{14}s_{34}} \right\} + \dots \quad (19)$$

$$A(1^+, 2^+, 3^-, 5^-, 4^+, 6^-) = \frac{\langle 3P \rangle^4}{s_{1P}s_{3P}s_{4P}} \left\{ \frac{[14]^2[23]^2}{\mathbf{s}^2(\mathbf{s}^2 s_{23} + \mathbf{c}^2 s_{14})s_{23}} + \frac{[12]^2[34]^2}{\mathbf{c}^2(\mathbf{s}^2 s_{34} + \mathbf{c}^2 s_{12})s_{12}} + \frac{[12][23][34][41]}{\mathbf{c}^2 \mathbf{s}^2 s_{12}s_{23}} \right\} + \dots \quad (20)$$

where we omitted terms that vanish in the $\epsilon \rightarrow 0$ limit; we also set $g^2 = \kappa = 1$. After substituting into Eq. (11) and using momentum conservation, we obtain

$$A(1^+, 2^+, 3^-, 4^+; P^{-2}) = \frac{\langle 3P \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}, \quad (21)$$

in agreement with Ref. [4]. For other NMHV helicity configurations, Eq. (11) follows in exactly the same way.

Acknowledgments

TRT is grateful to CERN Theory Unit and to Max-Planck-Institut für Physik in München, where substantial portions of this work were performed, for their hospitality and financial support. This material is based in part upon work supported by the National Science Foundation under Grant No. PHY-1314774. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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